Pearson Edexcel

Mark Scheme (Results)

Summer 2024

Pearson Edexcel GCE

In A Level Further Mathematics (9FM0)

Paper 4A Pure Mathematics

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

EDEXCEL GCE MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: method marks are awarded for ‘knowing a method and attempting to apply it’, unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\sqrt{}$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- ∗ The answer is printed on the paper
- ⌈ The second mark is dependent on gaining the first mark

4. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

5. Where a candidate has made multiple responses <u>and indicates which response they wish to submit</u>, examiners should mark this response.
If there are several attempts at a question <u>which have not been crossed out</u>, examiners should mark the final answer which is the answer that is the <u>most complete</u>.

6. Ignore wrong working or incorrect statements following a correct answer.

7. Mark schemes will firstly show the solution judged to be the most common response expected from candidates. Where appropriate, alternatives answers are provided in the notes. If examiners are not sure if an answer is acceptable, they will check the mark scheme to see if an alternative answer is given for the method used.

Question	Scheme	Marks	AOs
1	$21^{22} \equiv 1 \pmod{23}$ or $21^{23} \equiv 21 \pmod{23}$ or $21^{23} \equiv -2 \pmod{23}$	B1	1.2
	$80 = 3\times 22 + 14 \Rightarrow 21^{80} \equiv 21^{14} \pmod{23}$ $80 = 3\times 23 + 11 \Rightarrow 21^{80} \equiv -8\times 21^{11} \pmod{23}$	M1	1.1b
	For example $\equiv \left((-2)^4\right)^3 \times (-2)^2 \equiv 16^3 \times 4 \equiv (-7)^2 \times -7 \times 4 \equiv 49 \times -28 \equiv 3\times 18 \equiv 54$ $\equiv (-2)^{14} \pmod{23} \equiv 16384 \pmod{23}$ $\equiv (-2)^{14} \equiv (4)^7 \pmod{23}$ $\equiv \left((-2)^4\right)^3 \times (-2)^2 \equiv 16^3 \times 4 \equiv \{16384\}$ $21^2 = 441 \equiv 4 \pmod{23} \Rightarrow \left(21^2\right)^7 = (4)^7 = \{16384\}$ $(3\times 7)^{14} = 3^{14}\times 7^{14} = 3^{14}\times (49)^7 \equiv 4\times (3)^7 \pmod{23} \equiv 4\times 2 \pmod{23}$ $\equiv -8\times(-2)^{11} \equiv 16\times 2^{10}$ Leading to $\equiv \ldots \pmod{23}$	dM1	1.1b
	$\equiv 8 \pmod{23}$	A1	2.2a
		(4)	
(4 marks)			

Notes:

B1: Accurately recalls Fermat's Little Theorem. Allow if stated correctly in general form, $a^{p-1} \equiv 1 \pmod{p}$. May be implied.

M1: Attempts to write the index in a suitable form and then applies the theorem, look for $80 = 22k + r$ or equivalent leading to a reduced index.

dM1: For a full method to reduce to a least residue. There are many ways that they can split up their value, including use of a calculator to work with a higher power as long as an attempt at FLT to reduce initially has been seen. Condone a sign slip e.g using $21^2 \equiv 19 \pmod{23}$ instead of $21^2 \equiv -19 \pmod{23}$ as long as there is a complete method and there is only one sign slip

A1: Correct answer.

Question	Scheme	Marks	AOs
2	Aux equation is $r^2 - 6r + 9 = 0 \Rightarrow r = ...$	**M1**	1.1b
	$(r-3)^2 = 0 \Rightarrow r = 3$	**A1**	1.1b
	General form is $u_n = (A + Bn)3^n$	**M1**	1.1a
	$\begin{cases} 4 = 3(A+B) \\ 6 = 9(A+2B) \end{cases} \Rightarrow A = ..., B = ...$	**M1**	2.1
	$u_n = \left(2 - \frac{2}{3}n\right)3^n$ o.e $u_n = (6-2n)3^{n-1}$	**A1**	1.1b
		(5)	
			(5 marks)

Notes:

M1: Forms and solves the auxiliary equation.

A1: Correct single root of 3.

M1: Selects the correct form for u_n for their roots of the equation. (If distinct real roots were found allow for $u_n = A\alpha^n + B\beta^n$)

M1: Uses the values of u_1 and u_2 with the corresponding values of *n* used to form and uses a correct method to solve simultaneous equations to find the constants. If no method is shown, use of calculator, for solving the simultaneous equations the values must be correct for their equations.

A1: Correct closed form, isw

Question	Scheme	Marks	AOs
3(a)	$234 = 2\times 96 + 42$	M1	1.1b
	$96 = 2\times 42 + 12$; $42 = 3\times 12 + 6$; $12 = 2\times 6(+0)$	M1	1.1b
	Hence $h = 6$	A1	2.2a
		(3)	
(b)	Using back substitution $6 = 42 - 3\times 12$	M1	1.1b
	$= 42 - 3(96 - 2\times 42) = 7\times 42 - 3\times 96$ $= 7\times(234 - 2\times 96) - 3\times 96$	dM1	1.1b
	$= 7\times 234 - 17\times 96$ (so $a = 7$ and $b = -17$)	A1	1.1b
		(3)	
(c)	As 6 divides 36 the congruence is equivalent to $16x \equiv 6 \pmod{39}$	B1	2.2a
	From (b) we deduce $1 = 7\times 39 - 17\times 16$ so -17 (or 22) is a multiplicative inverse of 16 modulo 39 or $16x \equiv 6 \pmod{39} \Rightarrow 2\times 8x \equiv 2\times 3 \pmod{39} \Rightarrow 8x \equiv 3 \pmod{39}$ Finds multiplicative inverse of 8 e.g. $1 = 5\times 8 - 39$	M1	3.1a
	Hence $22\times 16x \equiv 22\times 6 \pmod{39}$ or $-17\times 16x \equiv -17\times 6 \pmod{39}$ or Hence $5\times 8x \equiv 5\times 3 \pmod{39}$ Leading to $x \equiv \ldots \pmod{39}$	M1	1.1b
	$x \equiv 132 \equiv 15 \pmod{39}$ Accept just 15	A1	1.1b
	So the solution is $x \equiv 15 \pmod{39}$ or $x \equiv 15, 54, 93, 132, 171$ or $210 \pmod{234}$	A1	2.3
		(5)	
		(11 marks)	

Notes:

(a)

M1: Starts the process of using the algorithm, with attempt at $234 = p\times 96 + q$.

M1: Continues the process until remainder zero is reached.

A1: Deduces the correct highest common factor from correct work.

(b)

M1: Begins the process of back substitution by rearranging their equation with least positive remainder.

dM1: Completes the process.

A1: Correct expression or values of *a* and *b* identified.

(c)

B1: Deduces the correct equivalent congruence.

M1: Chooses a suitable strategy to solve the reduced congruence. May use multiplicative inverse as per scheme or see alts for some variations, must be using their value for *b*

M1: Applies their multiplicative inverse to reach $x \equiv ...(\text{mod } 39)$

In the alternative it is for reaching $x \equiv ...(\text{mod } 39)$ from a succession of multiples.

A1: For 15 as a solution, need not have the $(\text{mod } 39)$. (Accept one correct solution if an alternative method is used.)

A1: For stating the solution is $15\ (\text{mod } 39)$ as the answer (not just in working) **or** for listing all the solutions modulo 234. Either way of expressing the answer is fine (single answer mod 39, or all 6 mod 234), but must realise it is more than just the 15, so do not accept just 15 with no indication of the modulus being considered.

ISW if they achieve the correct answer and then try to list solutions

SPECIAL CASE: B1 For one correct value of *x* found

Alts for (c)

(c) **Way 2**	From (b) we deduce $1 = 7\times 39 - 17\times 16$ so -17 (or 22) is a multiplicative inverse of 96 modulo 239	**B1**	2.2a
	Hence $-17\times 96x \equiv -17\times 36(\text{mod } 234)$ or $6x \equiv -612(\text{mod } 234)$	**M1**	3.1a
	As 6 divides the congruence $6x \equiv 90(\text{mod } 234)$ to reach $x \equiv ...(\text{mod } 39)$ As 6 divides the congruence $6x \equiv -612(\text{mod } 234)$ to reach $x \equiv ...(\text{mod } 39)$	**M1**	1.1b
	$x \equiv 15(\text{mod } 39)$ Accept just 15	**A1**	1.1b
	So the solution is $x \equiv 15(\text{mod } 39)$ or $x \equiv 15, 54, 93, 132, 171 \text{ or } 210\ (\text{mod } 234)$	**A1**	2.3
		(5)	
	Notes **B1:** Deduces the multiplicative inverse of 96 **M1:** Multiplies through by the multiplicative inverse of 96 **M1:** For using reaching $x \equiv ...(\text{mod } 39)$. **A1:** For 15 as a solution, need not have the $(\text{mod } 39)$. (Accept one correct solution if an alternative method is used.		

	A1: For stating the solution is $15 \pmod{39}$ as the answer (not just in working) or listing all the solutions modulo 234. As per main scheme do not accept just 15 with no indication of the modulus being considered.		
(c) **Way 3**	As 6 divides 36 the congruence is equivalent to $16x \equiv 6 \pmod{39}$	**B1**	2.2a
	$5 \times 16x \equiv 5 \times 6 \pmod{39} \Rightarrow 2x \equiv 30 \pmod{39}$	**M1**	3.1a
	$\Rightarrow 20 \times 2x \equiv 20 \times 30 \pmod{39} \Rightarrow 1x \equiv 15 \pmod{39}$	**M1**	1.1b
	$x \equiv 15 \pmod{39}$ Accept just 15	**A1**	1.1b
	So the solution is $x \equiv 15 \pmod{39}$ or $x \equiv 15, 54, 93, 132, 171$ or $210 \pmod{234}$	**A1**	2.3
		(5)	

Notes

B1: Deduces the correct equivalent congruence.

M1: Tries multiplying by various numbers in an attempt to reduce the coefficient to 1. One example shown above but others are possible. Allow the M for an attempt at starting such a process.

M1: For using reaching $x \equiv ... \pmod{39}$ from a succession of multiples.

A1: For 15 as a solution, need not have the $\pmod{39}$. (Accept one correct solution if an alternative method is used.

A1: For stating the solution is $15 \pmod{39}$ as the answer (not just in working) or listing all the solutions modulo 234. As per main scheme do not accept just 15 with no indication of the modulus being considered.

(c) **Way 4**	As 6 divides 36 the congruence is equivalent to $16x \equiv 6 \pmod{39}$	**B1**	2.2a
	So $16x \equiv 6, 45, 84, 123, ...$	**M1**	3.1a
	$16x \equiv ..., 240$	**M1**	1.1b
	$240 = 16 \times 15 \Rightarrow x \equiv 15 \pmod{39}$ Accept just 15	**A1**	1.1b
	So the solution is $x \equiv 15 \pmod{39}$ or $x \equiv 15, 54, 93, 132, 171$ or $210 \pmod{234}$	**A1**	2.3
		(5)	

Notes

B1: Deduces the correct equivalent congruence.

M1: Works out the possibilities for $16x$ to try and find one that is divisible by 16. Look for the first few evaluated.

M1: For reaching a value for $16x$ that is a multiple of 16.

A1: For 15 as a solution, need not have the $\pmod{39}$. (Accept one correct solution if an alternative method is used.

A1: For stating the solution is $15 \pmod{39}$ as the answer (not just in working) or listing all the solutions modulo 234. As per main scheme do not accept just 15 with no indication of the modulus being considered.

NB For students who try dividing though by 12 to get $8x \equiv 3 \pmod{234}$ a maximum of B0M1M0A0A0 is possible for attempt to reach $x \equiv ... \pmod{234}$ via valid method.

Question	Scheme	Marks	AOs
4(a)	$2\times4-1\times2+2\times0=\lambda\times2 \Rightarrow \lambda=...$ or $2\times0-1\times-2+2\times2=\lambda\times2 \Rightarrow \lambda=...$ **Alternative 1** $\begin{pmatrix}4&2&0\\2&p&-2\\0&-2&2\end{pmatrix}\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}6\\-p\\6\end{pmatrix}=\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and deduces a value for λ **Alternative 2** $\begin{pmatrix}4-\lambda&2&0\\2&p-\lambda&-2\\0&-2&2-\lambda\end{pmatrix}\begin{pmatrix}2\\-1\\2\end{pmatrix}=0 \Rightarrow \begin{matrix}2(4-\lambda)-2=0\\2+2(2-\lambda)=0\end{matrix} \Rightarrow \lambda=...$	**M1**	1.1b
	$\lambda=3$	**A1**	2.2a
		(2)	
(b)	$2\times2-p-2\times2=3\times-1 \Rightarrow 4-p-4=-3 \Rightarrow p=3*$ **Alternative 1** $(4-3)\left((p-3)(2-3)-4\right)-2\left(2(2-3)-0\right)+0=0 \Rightarrow p=3*$ **Using Alternative 2 from (a)** $4-(p-\lambda)-4=0$ uses $\lambda=3$ to show $p=3*$	**B1***	1.1b
		(1)	
(c)(i)	$(4-\lambda)\left((3-\lambda)(2-\lambda)-4\right)-2\left(2(2-\lambda)-0\right)+0=0$	**M1**	1.1b
	$\Rightarrow (4-\lambda)\left(\lambda^2-5\lambda+2\right)-8+4\lambda=0 \Rightarrow \lambda^3-9\lambda^2+18\lambda=0$ $\Rightarrow \lambda(\lambda-3)(\lambda-6)=0 \Rightarrow \lambda=...$	**M1**	1.1b
	Remaining eigenvalues are 0 and 6.	**A1**	1.1b
		(3)	
(ii)	$\left.\begin{matrix}4x+2y=0\\2x+3y-2z=0\\-2y+2z=0\end{matrix}\right\}$ or $\left.\begin{matrix}-2x+2y=0\\2x-3y-2z=0\\-2y-4z=0\end{matrix}\right\} \Rightarrow x=..,y=..,z=..$	**M1**	2.1
	$c\begin{pmatrix}-1\\2\\2\end{pmatrix}$ or $d\begin{pmatrix}2\\2\\-1\end{pmatrix}$ o.e.	**A1**	1.1b

	$c\begin{pmatrix}-1\\2\\2\end{pmatrix}$ and $d\begin{pmatrix}2\\2\\-1\end{pmatrix}$ o.e.	**A1**	1.1b
		(3)	
(d)	$\mathbf{D}=\begin{pmatrix}3&0&0\\0&6&0\\0&0&0\end{pmatrix}$ or $\mathbf{D}=\begin{pmatrix}3&0&0\\0&0&0\\0&0&6\end{pmatrix}$ or equivalent	**B1ft**	2.2a
	$\sqrt{2^2+(-1)^2+2^2}=\sqrt{9}=3\Rightarrow \mathbf{v}=...$	**M1**	3.1a
	$\mathbf{P}=\frac{1}{3}\begin{pmatrix}2&2&-1\\-1&2&2\\2&-1&2\end{pmatrix}$ or $\mathbf{P}=\frac{1}{3}\begin{pmatrix}2&-1&2\\-1&2&2\\2&2&-1\end{pmatrix}$ The order must correspond to their **D**	**A1ft**	1.1b
		(3)	
			(12 marks)

Notes:

(a)

M1: Uses either the first or third row of matrix with the eigenvector to form and solve an equation in λ. May see the full matrix equation used, but this is not necessary.

Alternative 1: multiples the matrix by the eigenvector and compares to a multiple of the eigenvector to deduce a value for λ

Alternative 2: Uses $\begin{pmatrix}4-\lambda&2&0\\2&p-\lambda&-2\\0&-2&2-\lambda\end{pmatrix}\begin{pmatrix}2\\-1\\2\end{pmatrix}=0$ to find an equation and find a value for λ

A1: For the correct eigenvalue of 3.

(b)

B1*: Uses the second row with eigenvector 3 to set up a correct equation in p and then solves correctly.

Alternative 1: finds the characteristic equations, use $\lambda=3$ and correctly shows that $p=3$

Alternative 2: Uses Alternative 2 from (a) and $\lambda=3$ to correctly show that $p=3$

(c)(i)

M1: Attempts the characteristic equation. Allow sign slips.

M1: Expands, simplifies and factorises to find the values, or equivalent method (e.g. solve by calculator).

A1: Correct remaining values.

(c)(ii)

M1: Correct method for one of the two required eigenvectors.

A1: One correct eigenvector. Allow any non-zero scalar multiple.

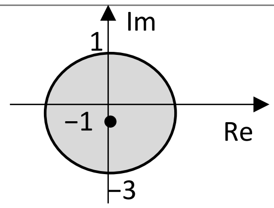
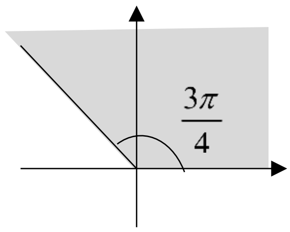
A1: Both correct eigenvectors. Allow any non-zero scalar multiple.

(d)

B1ft: Correct diagonal matrix, following through on their eigenvalues.

M1: Normalises the eigenvectors.

A1ft: Forms correct matrix **P** with columns in appropriate order for their **D**, following through on their eigenvectors.

Question	Scheme	Marks	AOs
5(i)(a)	$\lvert x+(y-3)\mathrm{i}\rvert = 2\lvert x+y\mathrm{i}\rvert \Rightarrow x^2+(y-3)^2 = 4\left(x^2+y^2\right)$	**M1**	3.1a
	$\Rightarrow x^2+y^2-6y+9 = 4x^2+4y^2 \Rightarrow x^2+y^2+2y-3=0$ in any order	**M1** **A1**	1.1b 1.1b
		(3)	
Alt (a)	Points are twice as far from 3i as from 0 so i and −3i are diametrically opposite points.	**M1**	3.1a
	So radius is $\dfrac{\lvert \mathrm{i}-(-3\mathrm{i})\rvert}{2} = 2$ and centre is $\dfrac{\mathrm{i}-3\mathrm{i}}{2} = -\mathrm{i}$	**M1**	1.1b
	Hence equation is $x^2+(y+1)^2=4$	**A1**	1.1b
		(3)	
(i)(b)	 Circle drawn with the inside shaded	**M1**	2.2a
	Correct circle drawn for their equation. Implied by the position of the centre and radius. Inside shaded.	**A1ft**	3.1a
		(2)	
(ii)(a)	A point z is mapped to a point with 3 times the argument… Rotate every point by 2 times the argument	**B1**	2.4
	… and with modulus as the modulus of the cube of z.	**B1**	2.5
		(2)	
(ii)(b)	 Sector from O along the real axis, indicated by some shading, in an anticlockwise direction	**M1**	1.1b
	Correct sector, angle must be stated or implied by the diagram	**A1**	2.2a
		(2)	
		(9 marks)	

Notes:

(i)(a)

M1: Substitutes $z = x + y\mathrm{i}$ into the equation and applies the modulus to obtain an equation with no i's. Must have dealt with the i^2 correctly. Condone not squaring the 2.

M1: Expands and gathers terms. Must have an x^2 and y^2 after simplifying so that it is an equation of a circle.

A1: Cancels common factor 3 to obtain the equation shown. ISW if they make errors trying to complete the square.

Note: Any letters may be used not just x and y

Alt

M1: Identifies two diametrically opposite points on the circle by understanding the geometry of the situation.
M1: Finds the radius and centre of the circle from their points, dotted or solid line
A1: Correct equation need not expand, but should be in simplest form.
(i)(b)
M1: Circle drawn anywhere and the inside shaded
A1ft: Correct circle drawn for their equation. Implied by the position of the centre and radius. Inside shaded.

(ii)(a)
B1: Identifies that the argument triples in size.
B1: Identifies that the modulus scales according to the modulus of z cubed.
(ii)(b)
M1: A sector centre O and starting along the real axis and in an anticlockwise direction. Must be some shading to represent the region, dotted or solid line
A1: Correct sector shaded and the angle stated or implied

Question	Scheme	Marks	AOs
6(a)	$I_{n+2} = \int \frac{\cos(n+2)x}{\sin x}\,\mathrm{d}x = \int \frac{\cos(nx)\cos 2x - \sin(nx)\sin 2x}{\sin x}\,\mathrm{d}x$	**M1**	3.1a
	$= \int \frac{\cos(nx)(1-2\sin^2 x) - 2\sin x\cos x\sin(nx)}{\sin x}\,\mathrm{d}x$	**M1** **A1**	1.1b 2.1
	$= \int \frac{\cos(nx)}{\sin x}\,\mathrm{d}x - \int \frac{\cos(nx)(2\sin^2 x) + 2\sin x\cos x\sin(nx)}{\sin x}\,\mathrm{d}x$ $= I_n - 2\int \cos(nx)\sin x + \cos x\sin(nx)\mathrm{d}x$	**dM1**	1.1b
	$= I_n - 2\int \sin(n+1)x\ \mathrm{d}x$	**ddM1**	2.2a
	$I_{n+2} = 2\frac{\cos(n+1)x}{n+1} + I_n$ *	**A1***	2.1
		(6)	
Alt 1	$I_n = \int \frac{\cos(n+1-1)x}{\sin x}\,\mathrm{d}x = \int \frac{\cos(n+1)x\cos x + \sin(n+1)\sin x}{\sin x}\,\mathrm{d}x$	**M1**	3.1a
	$= \int \frac{\cos(n+2)x + \cos nx}{2\sin x} + \sin(n+1)x\ \mathrm{d}x$	**M1** **A1**	1.1b 2.1
	$I_n = \frac{1}{2}I_{n+2} + \frac{1}{2}I_n + \int \sin(n+1)x\ \mathrm{d}x$	**dM1**	1.1b
	$I_n = \frac{1}{2}I_{n+2} + \frac{1}{2}I_n - \frac{1}{n+1}\ \cos(n+1)x\ \mathrm{d}x \Rightarrow I_{n+2} = \ldots$	**ddM1**	2.2a
	$I_{n+2} = 2\frac{\cos(n+1)x}{n+1} + I_n$ *	**A1***	2.1
		(6)	
Alt 2	$I_{n+2} = \int \frac{\cos(n+2)x}{\sin x}\,\mathrm{d}x = \int \frac{\cos(n+1)x\cos x - \sin(n+1)x\sin x}{\sin x}\,\mathrm{d}x$	**M1**	3.1a
	$= \int \frac{\cos(n+2)x + \cos nx}{2\sin x} - \sin(n+1)x\ \mathrm{d}x$ Or $= \int \frac{(\cos nx\cos x - \sin nx\sin x)\cos x}{\sin x} - \sin(n+1)x\ \mathrm{d}x$ $= \int \frac{\cos nx(1-\sin^2 x) - \sin nx\sin x\cos x}{\sin x} - \sin(n+1)x\ \mathrm{d}x$ $= \int \frac{\cos nx}{\sin x} - \cos nx\sin x - \sin nx\cos x - \sin(n+1)x\ \mathrm{d}x$ $= \int \frac{\cos nx}{\sin x} - 2\sin(n+1)x\ \mathrm{d}x$	**M1** **A1**	1.1b 2.1

	$I_{n+2}=\frac{1}{2}I_{n+2}+\frac{1}{2}I_n-\int\sin(n+1)x\,\mathrm{d}x$ Or $I_{n+2}=I_n-2\int\sin(n+1)x\,\mathrm{d}x$	**dM1**	1.1b
	$I_{n+2}=\frac{1}{2}I_{n+2}+\frac{1}{2}I_n+\frac{1}{n+1}\cos(n+1)x\,\mathrm{d}x\Rightarrow I_{n+2}=...$ Or $I_{n+2}=I_n+\frac{2}{n+1}\cos(n+1)x\,\mathrm{d}x\Rightarrow I_{n+2}=...$	**ddM1**	2.2a
	$I_{n+2}=2\frac{\cos(n+1)x}{n+1}+I_n$	**A1***	2.1
		(6)	
(b)	$[I_1]=\int\cot x\,\mathrm{d}x=[\ln\sin x]$ may be seen in an expression for I_5	**M1**	1.1b
	$=\ln\frac{\sqrt{3}}{2}-\ln\frac{\sqrt{2}}{2}$ may be seen in an expression for I_5	**A1**	2.2a
	$I_5=2\frac{\cos 4x}{4}+I_3$ or $I_3=2\frac{\cos 2x}{2}+I_1$	**M1**	1.1b
	$[I_5]_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\left[\frac{\cos 4x}{2}+\cos 2x+I_1\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ or $\left[\frac{\cos 4x}{2}+\cos 2x+\ln\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=\frac{1}{2}\cos\frac{4\pi}{3}+\cos\frac{2\pi}{3}-\frac{1}{2}\cos\pi-\cos\frac{\pi}{2}+\text{"}\ln\frac{\sqrt{3}}{2}-\ln\frac{\sqrt{2}}{2}\text{"}$	**M1**	1.1b
	$\left(=-\frac{1}{4}-\frac{1}{2}+\frac{1}{2}-0+\ln\frac{\sqrt{3}}{2}-\ln\frac{\sqrt{2}}{2}\right)=-\frac{1}{4}+\frac{1}{2}\ln\frac{3}{2}$ (oe)	**A1**	2.1
		(5)	
			(11 marks)

Notes:

(a)

M1: Applies the compound angle formula as shown.

M1: Applies both double angle formulae $\cos 2x=1-2\sin^2 x$ and $\sin 2x=2\sin x\cos x$ to the expression.

A1: Correct intermediate step reached, need not be simplified.

dM1: Dependent on previous method mark. Splits the integrand to identify I_n and cancels the sin*x* term in the remaining integral.

ddM1: Dependent on both previous method marks. Applies the compound angle formula to combine terms.

A1*: Correct completion to the given result. No errors seen.

Alternative 1

M1: Applies the compound angle formula as shown.

M1: Applies the sum product formula $\cos P\cos Q=\frac{1}{2}\left[\cos(P+Q)+\cos(P-Q)\right]$

A1: Correct intermediate step reached, need not be simplified.

dM1: Dependent on previous method mark. Splits the integrand to identify I_n and. I_{n+2}

ddM1: Dependent on both previous method marks. Integrates $\int \sin(n+1)x$ and rearranges to make I_{n+2}

A1*: Correct completion to the given result. No errors seen.

Alternative 2

M1: Applies the compound angle formula as shown.

M1: Applies the sum product formula $\cos P \cos Q = \frac{1}{2}[\cos(P+Q)+\cos(P-Q)]$

Alternatively uses $\cos(A+B) = \cos A\cos B - \sin A \sin B$, $\cos^2 x = 1-\sin^2 x$ and $\sin A\cos B + \cos A \sin B = \sin(A+B)$ in an attempt to simplify.

A1: Correct intermediate step reached, need not be simplified.

dM1: Dependent on previous method mark. Splits the integrand to identify I_n and. I_{n+2}

ddM1: Dependent on both previous method marks. Integrates $\int \sin(n+1)x$ and rearranges to make I_{n+2}

A1*: Correct completion to the given result. No errors seen.

(b)

M1: Attempts I_1, look for $K\ln\sin x$ anywhere in their solution

A1: A correct expression, not necessarily simplified, for I_1. May be seen as part of the final answer.

M1: Any one correct application of the reduction formula applied to the question (either working down or working up).

M1: Applies the reduction formula twice and substitutes the limits.

A1: Correct answer in the form required but accept equivalents in this form. E.g. accept $\dfrac{\ln\frac{9}{4}-1}{4}$ Logs must have been combined.

6(a) Alt 3	$I_{n+2} - I_n = \int \dfrac{\cos(n+2)x - \cos nx}{\sin x}\,dx$	**M1**	3.1a
	$= -\int \dfrac{2\sin\frac{1}{2}((n+2)x+nx)\sin\frac{1}{2}((n+2)x-nx)}{\sin x}\,dx$	**M1**	1.1b
	$= -\int \dfrac{2\sin(n+1)x\sin x}{\sin x}\,dx$	**A1**	2.1
	$= -2\int \sin(n+1)x\,dx$	**dM1**	1.1b
	$= 2\dfrac{\cos(n+1)x}{n+1} \Rightarrow I_{n+2} = \ldots$	**ddM1**	2.2a
	$\Rightarrow I_{n+2} = 2\dfrac{\cos(n+1)x}{n+1} + I_n$*	**A1***	2.1
		(6)	

Notes:

(a)

M1: Attempts the difference $I_{n+2} - I_n = \ldots$ and combines to a single fraction

M1: Applies the difference of two cosines terms formula.
A1: Correct expression.
dM1: Dependent on previous method mark. Cancels the sin$x$ term in the integral.
ddM1: Dependent on both previous method marks. Integrates.
A1*: Correct completion to the given result. No errors seen.

Question	Scheme	Marks	AOs
7(a)	$\mathbf{B}^2=\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\1&1\end{pmatrix}\{=\mathbf{E}\}$ Accept $\begin{pmatrix}2&1\\1&1\end{pmatrix}$ as long as $2\equiv 0(\text{mod}\,2)$ is used later. Or $\mathbf{B}^3=\begin{pmatrix}3&2\\2&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\{=\mathbf{I}\}$ Accept $\begin{pmatrix}3&2\\2&1\end{pmatrix}$ as long as $2\equiv 0(\text{mod}\,2)$ and $3\equiv 1(\text{mod}\,2)$ is used later.	M1	2.5
	$\mathbf{B}^3=\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\{=\mathbf{I}\}$ (and **B** is not the identity) hence **B** has order 3 or $\lvert\mathbf{B}\rvert=3$	A1	2.1
		(2)	
(b)	**I** {is the identity so} has order 1, and **E** {is $\mathbf{B}^{-1}$ so also} has order 3.	B1	1.1b
	Finds at least one of the remaining orders e.g. $\mathbf{A}^2=\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\{=\mathbf{I}\}$ or $\mathbf{C}^2=\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\{=\mathbf{I}\}$ or $\mathbf{D}^2=\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\{=\mathbf{I}\}$	M1	1.1b
	(So **I** has order 1, **B** and **E** order 3 and) **A**, **C** and **D** each have order 2	A1	1.1b
		(3)	
(c)	(i) There is no element of **order 6** in G {so cannot be isomorphic to C_6}	B1	2.4
	(ii) Either • There are 12 symmetries of a regular hexagon • Hexagon has more than 6 symmetries • Hexagon has 2 **rotations** of order 6, could give an example • Hexagon has 2 **rotations** of order 3 • Hexagon has 1 **rotation** of order 2 • Hexagon has 6 **reflections** of order 2 • Group of symmetries of a hexagon has an order greater than 6 B0 for • The group of symmetries of a hexagon is also cyclic • The groups are not of the same order • Any incorrect orders of element stated • Comments on the order of the whole group • The group of symmetries has order 6	B1	2.4
		(2)	
(d)	Any two correct matchings	M1	2.2a
	Completes the matching with at least 4 correct	M1	3.1a

Hence an isomorphism is

I	A	B	C	D	E
e	c	a	d	f	b
e	d	a	f	c	b
e	f	a	c	d	b

I	A	B	C	D	E
e	c	b	f	d	a
e	d	b	c	f	a
e	f	b	d	c	a

A1 2.1

(3)

(10 marks)

Notes:

(a)

M1: Squares **B** to begin the process of finding the order, or uses a calculator to find $\mathbf{B}^3$

A1: Shows that $\mathbf{B}^3 = \mathbf{I}$ and concludes that order must be 3.

(b)

B1: States the order for **I** and gives the order for **E** (which is the inverse of **B**).

M1: Finds the order of at least one of the other elements to get an element of order 2. May use a calculator

A1: Correct orders for the three elements of order 2.

(c)

B1: Correct reason given, identifies there is no element of **order 6.** It is not sufficient to say that it does not have a group generator, they need to give a reason.

B1: Any correct reason given.

Considers the total number of elements of the group, order of the group is more than 6

Considers the orders of either reflections or rotations

(d)

M1: Any two correct matchings

M1: Completes the matching with at least 4 correct

A1: A fully specified correct isomorphism.

Question	Scheme	Marks	AOs
8(a)	$\text{S.A.} = 2\pi\int_0^9 y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x; = 2\pi\int_0^9\left(\frac{9}{2}\mathrm{e}^{\frac{1}{9}x}\right)\sqrt{1+\left(\frac{1}{2}\mathrm{e}^{\frac{1}{9}x}\right)^2}\,\mathrm{d}x = \ldots$	**B1;** **M1**	1.1a 3.4
	$= 9\pi\int_0^9 \mathrm{e}^{\frac{1}{9}x}\sqrt{1+\frac{1}{4}\mathrm{e}^{\frac{2}{9}x}}\,\mathrm{d}x = \frac{9\pi}{2}\int_0^9 \mathrm{e}^{\frac{1}{9}x}\sqrt{4+\mathrm{e}^{\frac{2}{9}x}}\,\mathrm{d}x$	**A1**	2.1
		(3)	
(b)	$u = \mathrm{e}^{\frac{1}{9}x} \Rightarrow \mathrm{d}u = \frac{1}{9}\mathrm{e}^{\frac{1}{9}x}\mathrm{d}x$	**B1**	2.2a
	$\Rightarrow \int \mathrm{e}^{\frac{1}{9}x}\sqrt{4+\mathrm{e}^{\frac{2}{9}x}}\,\mathrm{d}x = \int \mathrm{e}^{\frac{1}{9}x}\sqrt{4+\mathrm{e}^{\frac{2}{9}x}}\,9\mathrm{e}^{-\frac{1}{9}x}\,\mathrm{d}u = 9\int\sqrt{4+u^2}\,\mathrm{d}u$	**M1**	1.1b
	$= 9\int\frac{4+u^2}{\sqrt{4+u^2}}\mathrm{d}u = 9\int\frac{2+u^2}{\sqrt{4+u^2}}\mathrm{d}u + 9\int\frac{2}{\sqrt{4+u^2}}\mathrm{d}u$ Or $= 9\int\frac{4+u^2}{\sqrt{4+u^2}}\mathrm{d}u = 9\int\frac{4u+u^3}{\sqrt{4u^2+u^4}}\mathrm{d}u$	**M1**	3.1a
	$= 9\int\frac{2u+u^3}{u\sqrt{4+u^2}}\mathrm{d}u + 18\int\frac{1}{\sqrt{4+u^2}}\mathrm{d}u$ Or $= 9\int\frac{4u+u^3}{\sqrt{4u^2+u^4}}\mathrm{d}u = 9\int\frac{2u+u^3}{\sqrt{4u^2+u^4}}\mathrm{d}u + 9\int\frac{2u}{\sqrt{4u^2+u^4}}\mathrm{d}u$	**M1**	2.1
	$\{x=0\Rightarrow\}\,u=1\,;\ \{x=9\Rightarrow\}\,u=\mathrm{e}$ $\Rightarrow \int_0^9 \mathrm{e}^{\frac{1}{9}x}\sqrt{4+\mathrm{e}^{\frac{2}{9}x}}\,\mathrm{d}x = 9\int_1^{\mathrm{e}}\frac{2u+u^3}{\sqrt{4u^2+u^4}}\mathrm{d}u + 18\int_1^{\mathrm{e}}\frac{1}{\sqrt{4+u^2}}\mathrm{d}u$ *	**A1***	3.4
		(5)	
(c)	$9\int\frac{2u+u^3}{\sqrt{4u^2+u^4}}\mathrm{d}u = \frac{9}{4}\int\frac{8u+4u^3}{\sqrt{4u^2+u^4}}\mathrm{d}u = \alpha\sqrt{4u^2+u^4}$	**M1**	1.1b
	$18\int\frac{1}{\sqrt{4+u^2}}\mathrm{d}u = \beta\,\mathrm{arsinh}\left(\frac{u}{2}\right)$ or $\beta\ln\left(u+\sqrt{u^2+a^2}\right)$	**M1**	1.1b
	$9\int_1^{\mathrm{e}}\frac{2u+u^3}{\sqrt{4u^2+u^4}}\mathrm{d}u + 18\int_1^{\mathrm{e}}\frac{1}{\sqrt{4+u^2}}\mathrm{d}u = \left[\frac{9}{2}\sqrt{4u^2+u^4} + 18\,\mathrm{arsinh}\left(\frac{u}{2}\right)\right]_1^{\mathrm{e}}$	**A1**	2.1

	Surface area $=\dfrac{9\pi}{2}\displaystyle\int_0^9 \mathrm{e}^{\frac{1}{9}x}\sqrt{4+\mathrm{e}^{\frac{2}{9}x}}\,\mathrm{d}x$ $=\dfrac{9\pi}{2}\left(\dfrac{9}{2}\sqrt{4\mathrm{e}^2+\mathrm{e}^4}+18\,\mathrm{arsinh}\left(\dfrac{\mathrm{e}}{2}\right)-\dfrac{9}{2}\sqrt{4+1}-18\,\mathrm{arsinh}\left(\dfrac{1}{2}\right)\right)=\ldots$ $=\dfrac{9\pi}{2}(42.6089\ldots)$	**ddM1**	3.4
	awrt 602 **cm²**	**A1**	3.2a
		(5)	
			(13 marks)

Notes:

(a)

B1: States or uses a correct formula for the surface area.

M1: Applies the surface area formula with $\dfrac{\mathrm{d}y}{\mathrm{d}x}=K\mathrm{e}^{\frac{1}{9}x}$

A1: Simplifies correctly as shown, so $K=\dfrac{9\pi}{2}$

(b)

B1: For a correct connecting equation between du and dx, accept any form, so $\mathrm{d}u=\dfrac{1}{9}\mathrm{e}^{\frac{1}{9}x}\mathrm{d}x$ or $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{9}\mathrm{e}^{\frac{1}{9}x}$ or any equivalent.

M1: Makes a full substitution into the equation to obtain an integral in terms of u only. Must have replaced the dx appropriately. Limit not needed.

M1: Writes the integral as $9\displaystyle\int\dfrac{4+u^2}{\sqrt{4+u^2}}\,\mathrm{d}u$ and then either

Splits the integral to extract a $\dfrac{1}{\sqrt{4+u^2}}$ term. Allow if the term is not the correct one.

Or

Multiplies through numerator and denominator in appropriate term to get the numerator.

M1: Multiplies through numerator and denominator in appropriate term to get the numerator of first integral correct.

Or

Splits the integral to extract a $\dfrac{u}{\sqrt{4u^2+u^4}}$ term

A1: Deduces the correct limits and achieves the given answer from fully correct work – must have split the integral correctly.

(c)

M1: Integrates to the correct form $9\displaystyle\int\dfrac{2u+u^3}{\sqrt{4u^2+u^4}}\,\mathrm{d}u=\alpha\sqrt{4u^2+u^4}$

M1: : Integrates to the correct form $18\int \frac{1}{\sqrt{4+u^2}}\,\mathrm{d}u = \beta\,\text{arsinh}\left(\frac{u}{2}\right)$ or $\beta\ln\left(u+\sqrt{u^2+a^2}\right)$

A1: Fully correct integration. Need not be simplified, and may be in terms of a different variable if a substitution was used. No need for constant of integration. No need for their $\frac{9\pi}{2}$ for this mark

ddM1: Completes the method to find the required surface area. Applies their limits from (b) to their integrated expression in *u* and subtracts the correct way round. If they have returned to an integral in *x* then limits 9 and 0 must be used. If they have changed variable again, then correct limits for their variable(s). They need their $\frac{9\pi}{2}$ for this mark. If they don't show the substitution of limits, follow through on

their $\frac{9\pi}{2}\times 42.6089...$ or a correct answer.

A1: awrt 602, units required.

Answer only, use of calculator with no integration shown scores no marks.

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